

## ANISOTROPIC TENSOR CALCULUS

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ABSTRACT. We introduce the anisotropic tensor calculus, which is a way of handling with tensors that depend on the direction remaining always in the same class. This means that the derivative of an anisotropic tensor is a tensor of the same type. As an application we show how to define derivations using anisotropic linear connections in a manifold. In particular, we show that the Chern connection of a Finsler metric can be interpreted as the Levi-Civita connection. Moreover, we also introduce the concept of anisotropic Lie derivative.

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## 1. INTRODUCTION

Anisotropic tensors appear in situations in that there is a strong dependence on the direction as it is the case of Finsler metrics and sprays. Traditionally, derivations of these tensors in a manifold  $M$  have been handled with linear connections in the vertical subbundle of  $TTM$  over the slit tangent bundle  $TM \setminus 0$ . This has the following drawback: you need to carry some information about derivatives which is not geometrically relevant. The most apparent consequence of this approach is that there are many linear connections which can be associated with a Finsler metric, loosing the unicity of the Levi-Civita connection in Riemannian geometry. Our aim is to solve these limitations by considering an anisotropic linear connection  $\nabla$ , which is a connection in the manifold but with a dependence on the direction (see Definition 2.10). Then when we compute the derivative of an anisotropic tensor  $T$  in  $v \in TM \setminus 0$ , we choose an extension  $V \in \mathfrak{X}(M)$  of  $v$  and make the computation with the affine linear connection  $\nabla^V$  and the tensor  $T_V$  (fixing the direction  $V(p)$  at every point  $p$  of the manifold  $M$ ). In principle, the result depends on the extension  $V$ , but this dependence disappears when a suitable vertical derivative of the tensor is subtracted (see (4)). This procedure has been inspired by the interpretation of the Chern connection as a family of affine connections, which was first developed in [7]. Later, in [3, 4], the relation of the flag curvature of the Finsler metric with the family of affine connections was completely clarified, setting down the foundations of anisotropic geometry. The central result of this paper is Theorem 2.7, which allows us to deal with and to develop the anisotropic tensor calculus departing from an anisotropic connection (which is a concept known in literature [9, Chapter 7]). The other central concept is the curvature tensor of an anisotropic derivative introduced in Lemma 2.11, which has already appeared implicitly for the Chern connection in [4, Theorem 2.1].

One of the main virtues of this approach is that the treatment of Finsler Geometry parallels the classical one of Riemannian Geometry using the Levi-Civita

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connection. In fact, with our anisotropic tensor calculus, the problem of unicity for connections associated with Finsler metrics and sprays is completely solved. As a matter of fact, given a linear connection on the vertical subbundle, we can construct an anisotropic linear connection on the manifold (see §2.8). It turns out that the four classical linear connections associated with a Finsler metric: Berwald, Cartan, Chern and Hashiguchi, project into only two anisotropic connections. The Berwald and Hashiguchi connections project into a connection that is the most natural one to be associated with a spray (see §2.7), whereas the Chern and Cartan ones project into an anisotropic connection which is the Levi-Civita connection of a Finsler metric, since it is the unique anisotropic connection which is torsion-free and such that  $\nabla g = 0$ , namely, the derivative of the fundamental tensor is trivial (see §2.6). Another remarkable fact is that we do not need to fix the nonlinear connection in order to obtain the invariants associated with a Finsler metric or a spray. All the invariants are obtained directly with the anisotropic connection differently from what happens when you consider a connection in the vertical subbundle, where the nonlinear connection is required to compute the geometric invariants.

Our last achievement is to introduce the anisotropic Lie derivative. This theory has been studied in anisotropic geometry for a long time [11], but without a clear tensorial development as in Riemannian geometry. The anisotropic approach allows one to define the Lie derivative of any anisotropic object by using the Lie bracket of vector fields as an anisotropic derivation (see Definition 2.5). There is another approach that uses  $TTM$  to define Lie derivatives, which implies more computations (see for example [6]). In Proposition 2.24, we obtain a very natural interpretation of the Lie derivative in terms of the flow of a vector field.

The paper is organized as follows. In Section 2, we introduce the anisotropic tensor calculus, the main result being Theorem 2.7, which ensures the existence of a unique anisotropic tensor derivation for every anisotropic derivation as in Definition 2.5. In §2.3 we introduce the notion of spray and its associated nonlinear connection. In §2.4 we give the definition of anisotropic linear connection and the curvature tensor associated. In §2.5, we introduce the notion of anisotropic covariant derivation and then of geodesic. We also establish the Jacobi equation of a geodesic in Proposition 2.19 and give a condition in terms of the difference tensor (see Remark 2.17) which implies that two different anisotropic connections determine the same Jacobi operator. In §2.6 and §2.7 we define the Chern and Berwald connections, showing that the first one is the Levi-Civita connection of a Finsler metric and the second one is the most natural connection associated with a spray. Some classical tensors are also introduced to describe the difference tensor between them. In §2.8 it is shown how to associate an anisotropic connection to a connection in the vertical subbundle. Finally, in §2.9, we show how to define the anisotropic Lie derivative using the Lie bracket and Theorem 2.7. We conclude with an interpretation of the anisotropic Lie derivative using the flow of a vector field (see Proposition 2.24).

## 2. ANISOTROPIC TENSOR CALCULUS

**2.1. Anisotropic tensors.** Let  $M$  be a manifold of dimension  $n$ ,  $TM$  its tangent bundle,  $TM^*$  is the cotangent bundle and  $A \subset TM \setminus 0$ , an open subset such that  $\pi(A) = M$ . Using the restriction of the natural projection  $\pi|_A : A \rightarrow M$  we can lift the bundle  $\pi : TM \rightarrow M$  to  $A$  obtaining a bundle  $\pi_A^*(M)$  and by lifting

$\pi : TM^* \rightarrow M$  we get another bundle  $\pi_A^*(M)^*$ . Then we can also make tensor products of these bundles as

$$\pi_A^*(M)^r \otimes (\pi_A^*(M)^*)^s := \pi_A^*(M) \otimes \overbrace{\cdots}^r \otimes \pi_A^*(M) \cdots \pi_A^*(M)^* \otimes \overbrace{\cdots}^s \otimes \pi_A^*(M)^*.$$

An  $A$ -anisotropic  $(r, s)$ -tensor in  $M$  is a smooth section of  $\pi_A^*(M)^r \otimes (\pi_A^*(M)^*)^s$ . The space of all the  $A$ -anisotropic  $(r, s)$ -tensors of  $M$  will be denoted by  $\mathfrak{T}_s^r(M, A)$ . Given a system of coordinates  $(\Omega, \varphi)$  in  $M$  with  $\varphi = (x^1, x^2, \dots, x^n)$ , if we denote by  $\partial_i$  the partial vector fields in  $\Omega$  and  $dx^i$  their dual one-forms, then for every  $T \in \mathfrak{T}_s^r(M, A)$  and every  $v \in A$  such that  $\pi(v) \in \Omega$ , we have that

$$T_v = \sum_{\substack{1 \leq i_1, i_2, \dots, i_r \leq n \\ 1 \leq j_1, j_2, \dots, j_s \leq n}} T_{i_1 i_2 \dots i_r}^{j_1 j_2 \dots j_s}(v) \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}.$$

Observe that at every point  $p \in M$ , we can identify  $T_p M$  with the dual vector space of  $T_p M^*$  and then the elements in the fibers of  $\pi_A^*(M)^r \otimes (\pi_A^*(M)^*)^s$  can be identified with multilinear maps

$$T : (T_p M^*)^r \times (T_p M)^s \rightarrow \mathbb{R}.$$

With this interpretation,  $T_{i_1 i_2 \dots i_r}^{j_1 j_2 \dots j_s}(v) = T_v(dx^{i_1}, dx^{i_2}, \dots, dx^{i_r}, \partial_{j_1}, \partial_{j_2}, \dots, \partial_{j_s})$ . We can also identify  $T \in \mathfrak{T}_s^r(M, A)$  with the  $\mathcal{F}(A)$ -multilinear map

$$T : \mathfrak{T}_1^0(M, A)^r \times \mathfrak{T}_0^1(M, A)^s \rightarrow \mathcal{F}(A), \quad (1)$$

where  $\mathcal{F}(A) = \{f : A \rightarrow \mathbb{R} : f \in C^\infty\}$ . In fact, every  $\mathcal{F}(A)$ -multilinear map as above determines a tensor  $T \in \mathfrak{T}_s^r(M, A)$ . In the case, we have a  $\mathcal{F}(A)$ -multilinear map  $T : \mathfrak{T}_0^1(M, A)^s \rightarrow \mathfrak{T}_0^1(M, A)$ , we define  $\bar{T} : \mathfrak{T}_1^0(M, A) \times \mathfrak{T}_0^1(M, A)^s \rightarrow \mathcal{F}(A)$  by

$$\bar{T}(\theta, X_1, \dots, X_s) = \theta(T(X_1, \dots, X_s)),$$

which is a  $(1, s)$ -tensor field. We shall consider  $T$  to be a tensor field itself, using the formula above only when necessary.

Finally, observe that when we fix an  $A$ -admissible vector field  $V$  in an open subset  $\Omega \subset M$ , then for every  $T \in \mathfrak{T}_s^r(M, A)$  we can define a tensor  $T_V \in \mathfrak{T}_s^r(\Omega)$  as

$$T_V : \mathfrak{X}^*(\Omega)^r \times \mathfrak{X}(\Omega)^s \rightarrow \mathcal{F}(\Omega)$$

such that  $T_V(\theta^1, \dots, \theta^r, X_1, \dots, X_s)(p) = T_{V(p)}(\theta^1(p), \dots, \theta^r(p), X_1(p), \dots, X_s(p))$ . The dependence on directions makes necessary to define derivatives in the vertical bundle.

**Definition 2.1.** Given a tensor  $T \in \mathfrak{T}_s^r(M, A)$ , we define its *vertical derivation* as the tensor  $\partial^\nu T \in \mathfrak{T}_s^{r+1}(M, A)$  given by

$$(\partial^\nu T)_v(Z, \theta^1, \theta^2, \dots, \theta^r, X_1, \dots, X_s) = \frac{\partial}{\partial t} T_{v+tZ(\pi(v))}(\theta^1, \theta^2, \dots, \theta^r, X_1, \dots, X_s)|_{t=0}$$

for any  $(Z, \theta^1, \theta^2, \dots, \theta^r, X_1, \dots, X_s) \in \mathfrak{T}_1^0(M, A)^{r+1} \times \mathfrak{T}_0^1(M, A)^s$ .

Like in the classical case of tensors on a manifold, we can define tensor products: if  $T_1 \in \mathfrak{T}_s^r(M, A)$  and  $T_2 \in \mathfrak{T}_{s'}^{r'}(M, A)$  then  $T_1 \otimes T_2 \in \mathfrak{T}_{s+s'}^{r+r'}(M, A)$  and it is given by

$$\begin{aligned} (T_1 \otimes T_2)(\theta^1, \dots, \theta^{r+r'}, X_1, \dots, X_{s+s'}) \\ = T_1(\theta^1, \dots, \theta^r, X_1, \dots, X_s) T_2(\theta^{r+1}, \dots, \theta^{r+r'}, X_{s+1}, \dots, X_{s+s'}). \end{aligned}$$

When both  $r, s \geq 1$ , we also can define the contractions.

**Definition 2.2.** Given an  $(r, s)$ -tensor  $T$ , its  $(i, j)$ -contraction is defined as the  $(r-1, s-1)$ -tensor  $\mathbf{C}_j^i(T)$  such that in a system of coordinates  $(\Omega, \varphi)$  as above, its coordinates are given by

$$\mathbf{C}_j^i(T)_{i_1 i_2 \dots i_{r-1}}^{j_1 j_2 \dots j_{s-1}}(v) = \sum_{m=1}^n T_{i_1 i_2 \dots i_{r-1} m i_i \dots i_{r-1}}^{j_1 j_2 \dots j_{s-1} m j_j \dots j_{s-1}}(v).$$

## 2.2. Tensor derivations.

**Definition 2.3.** A tensor derivation  $\mathcal{D}$  is the subset of  $\mathbb{R}$ -linear maps

$$\mathcal{D} : \mathfrak{T}_s^r(M, A) \rightarrow \mathfrak{T}_s^r(M, A)$$

for any  $r \geq 0, s \geq 0$ , such that for any tensors  $T_1$  and  $T_2$ :

$$\mathcal{D}(T_1 \otimes T_2) = \mathcal{D}T_1 \otimes T_2 + T_1 \otimes \mathcal{D}T_2$$

and  $\mathcal{D}(\mathbf{C}(T)) = \mathbf{C}(\mathcal{D}(T))$  for any tensor  $T$  and any contraction  $\mathbf{C}$ .

It is easy to check that tensor derivations are local in the sense that if we consider an open subset  $\mathcal{A}$  of  $A$ , then  $\mathcal{D}(T|_{\mathcal{A}}) = (\mathcal{D}T)|_{\mathcal{A}}$  and it obeys the product rule

$$\begin{aligned} \mathcal{D}(T(\theta^1, \dots, \theta^r, X_1, \dots, X_s)) &= (\mathcal{D}T)(\theta^1, \dots, \theta^r, X_1, \dots, X_s) \\ &+ \sum_{i=1}^r T(\theta^1, \dots, \mathcal{D}\theta^i, \dots, \theta^r, X_1, \dots, X_s) \\ &+ \sum_{j=1}^s T(\theta^1, \dots, \theta^r, X_1, \dots, \mathcal{D}X_j, \dots, X_s), \end{aligned}$$

where we have interpreted  $T$  as in (1). This can be proved in an analogous way to the classical case (see for example [8, Prop. 2.12 and 2.13]).

In the following, we will consider sometimes  $\mathfrak{X}(M)$  as a subset of  $\mathfrak{T}_0^1(M, A)$ , namely, a vector field  $X \in \mathfrak{X}(M)$  can be lifted to  $X^* \in \mathfrak{T}_0^1(M, A)$  by making  $X^*(v) = X(\pi(v))$ , and we will identify  $X^* \equiv X$ .

**Proposition 2.4.** A tensor derivation is determined by its value on  $\mathfrak{T}_0^0(M, A) = \mathcal{F}(A)$  and on  $\mathfrak{X}(M) \subset \mathfrak{T}_0^1(M, A)$ .

*Proof.* As a consequence of the product rule, the tensor derivation is determined by its value on  $\mathfrak{T}_0^1(M, A)$ ,  $\mathfrak{T}_1^0(M, A)$  and  $\mathfrak{T}_0^0(M, A)$ . Moreover, locally, in a system of coordinates  $(\Omega, \varphi)$ , any  $X \in \mathfrak{T}_0^1(M, A)$  can be expressed as  $X = \sum_{i=1}^n a^i \partial_i$ , with  $a_i \in \mathcal{F}(T\Omega \cap A)$ , and for  $\theta \in \mathfrak{T}_1^0(M, A)$ , we have

$$\mathcal{D}\theta(X) = \mathcal{D}(\theta X) - \theta(\mathcal{D}(X)). \quad (2)$$

□

**Definition 2.5.** Let us define an *anisotropic derivation*  $\delta$  in  $A$  with associated vector field  $Z$  as a map

$$\delta : A \times \mathfrak{X}(M) \rightarrow TM, \quad (v, X) \mapsto \delta^v X \in T_{\pi(v)}M,$$

such that

- (i)  $\delta^v(X + Y) = \delta^v X + \delta^v Y$ ,  $X, Y \in \mathfrak{X}(M)$ ,
- (ii)  $\delta^v(fX) = Z(f)X + f\delta^v X$  for any  $f \in \mathcal{F}(M)$ ,  $X \in \mathfrak{X}(M)$
- (iii)  $(\delta^V X)(p) := \delta^{V(p)} X$  is smooth for any  $V \in \mathfrak{X}(\Omega)$  with  $\Omega \subset M$ ,  $X \in \mathfrak{X}(M)$  and  $V$ ,  $A$ -admissible, that is,  $V(p) \in A$  for every  $p \in \Omega$ .

In the following, when working in the tangent bundle  $TM$ , we will use the natural coordinates  $(\Omega \times \mathbb{R}^n, \tilde{\varphi})$  associated with a coordinate system  $(\Omega, \varphi)$  of  $M$ . As usual, we will denote the coordinates of a point  $v \in TM$  as

$$\tilde{\varphi}^{-1}(v) = (x, y) = (x^1, x^2, \dots, x^n, y^1, y^2, \dots, y^n).$$

We will also use the *Einstein summation convention* when possible.

**Lemma 2.6.** *Given an anisotropic derivation  $\delta$  in  $A$  with associated vector field  $Z$ , we can define the derivation of a function  $h \in \mathcal{F}(A)$  as*

$$\mathcal{D}(h)(v) := Z(h(V))(\pi(v)) - \partial^\nu h(\delta^\nu V) \quad (3)$$

with  $V$  any vector field such that  $V(\pi(v)) = v$ .

*Proof.* The only non-trivial point is to check that  $\mathcal{D}(h)$  is well-defined, namely, it does not depend on the extension  $V$ . Consider a system of coordinates  $(\Omega, \varphi)$  and define the non-linear coefficients of  $\delta$  as the functions  $\delta_j^k(v)$  satisfying that  $\delta^v \partial_i = \delta_i^k(v) \partial_k$ . Then

$$\begin{aligned} \mathcal{D}(h) &= Z(h(V)) - \partial^\nu h(\delta^\nu V) \\ &= Z^i \frac{\partial h}{\partial x^i} + Z^i \frac{\partial V^j}{\partial x^i} \frac{\partial h}{\partial y^j} - Z^i \frac{\partial V^j}{\partial x^i} \frac{\partial h}{\partial y^j} - V^j \delta_j^k(v) \frac{\partial h}{\partial y_k} \\ &= Z^i \frac{\partial h}{\partial x^i} - v^j \delta_j^k(v) \frac{\partial h}{\partial y_k}, \end{aligned}$$

which concludes that  $\mathcal{D}(h)$  does not depend on the considered extension  $V$ .  $\square$

**Theorem 2.7.** *Let  $\delta$  be an anisotropic derivation in  $A \subset TM$  with associated vector field  $Z \in \mathfrak{X}(M)$ . Then there exists a unique tensor derivation  $\mathcal{D}$  such that  $\mathcal{D}(X) = \delta^v X$  and  $\mathcal{D}(h)$  is defined by (3).*

*Proof.* Recall that the derivation of a one-form is given by (2) and the tensor derivation of a tensor  $T \in \mathfrak{T}_s^r(M, A)$  by

$$\begin{aligned} (\mathcal{D}T)(\theta^1, \dots, \theta^r, X_1, \dots, X_s) &= \mathcal{D}(T(\theta^1, \dots, \theta^r, X_1, \dots, X_s)) \\ &\quad - \sum_{i=1}^r T(\theta^1, \dots, \mathcal{D}\theta^i, \dots, \theta^r, X_1, \dots, X_s) \\ &\quad - \sum_{j=1}^s T(\theta^1, \dots, \theta^r, X_1, \dots, \mathcal{D}X_j, \dots, X_s). \end{aligned}$$

Moreover, it is straightforward that

$$\begin{aligned} \mathcal{D}(T(\theta^1, \dots, \theta^r, X_1, \dots, X_s)) \\ = Z(T_V(\theta^1, \dots, \theta^r, X_1, \dots, X_s)) - \partial^\nu T(\delta^\nu V, \theta^1, \dots, \theta^r, X_1, \dots, X_s) \quad (4) \end{aligned}$$

and then

$$\begin{aligned}
(\mathcal{D}T)(\theta^1, \dots, \theta^r, X_1, \dots, X_s) &:= Z(T_V(\theta^1, \dots, \theta^r, X_1, \dots, X_s)) \\
&\quad - \partial^\nu T(\delta^\nu V, \theta^1, \dots, \theta^r, X_1, \dots, X_s) \\
&\quad - \sum_{i=1}^r T(\theta^1, \dots, \mathcal{D}\theta^i, \dots, \theta^r, X_1, \dots, X_s) \\
&\quad - \sum_{j=1}^s T(\theta^1, \dots, \theta^r, X_1, \dots, \mathcal{D}X_j, \dots, X_s).
\end{aligned}$$

A computation in coordinates as in the proof of Lemma 2.6 shows that the definition does not depend on the extension  $V$ . Moreover, it follows easily that  $\mathcal{D}(fg) = \mathcal{D}(f)g + f\mathcal{D}(g)$  and then as in the classical case of tensor derivations (see [8, Theorem 2.15]) that  $\mathcal{D}$  is an anisotropic tensor derivation (it commutes with contractions and it satisfies the leibnitzian rule for the tensor product).  $\square$

**2.3. Nonlinear connection of a spray.** Let us introduce some general notation. Given a manifold  $M$ , we will denote by  $\pi_M : TM \rightarrow M$  the natural projection from its tangent bundle. In particular,  $\pi_{TM} : TTM \rightarrow TM$  denotes the natural projection from the tangent bundle  $TTM$  of  $TM$ . Recall that the vertical distribution of  $TM$  is defined as  $\mathcal{V}TM = \ker(\pi_{TM})$  (we will also use  $\mathcal{V}_v TM = \mathcal{V}TM \cap \pi_{TM}^{-1}(v)$ ). Moreover, we will denote by  $C$  the *canonical vertical vector field* defined as  $C(v) = i_v(v)$ , where

$$i_v : T_{\pi(v)}M \rightarrow \mathcal{V}_v TM \quad (5)$$

is given by  $i_v(u) = \frac{d}{dt}(v + tu)|_{t=0}$ .

**Definition 2.8.** Given a manifold  $M$ , a *semi-spray* on  $M$  is a vector field  $S$  in a subset  $A \subset TM$  which has the following property:

if  $\beta : [a, b] \subset \mathbb{R} \rightarrow A$  is an integral curve of  $S$ , then  $\beta = \dot{\alpha}$ , where  $\alpha = \pi_M \circ \beta$ . (6)

Condition (6) is equivalent to say that  $d\pi_M(S) = \pi_{TM}(S)$ . Moreover, in natural coordinates  $(\tilde{\varphi}, \Omega \times \mathbb{R}^n)$  for  $TM$  with  $\tilde{\varphi} = (x^1, \dots, x^n, y^1, \dots, y^n)$  associated with a coordinate system  $(\varphi = (x^1, \dots, x^n), \Omega)$  of  $M$ ,  $S$  is a semi-spray if and only if it is expressed as

$$S(v) = \sum_{i=1}^n (y^i \frac{\partial}{\partial x^i} - G^i(v) \frac{\partial}{\partial y^i}) \quad (7)$$

where  $G^i : (\Omega \times \mathbb{R}^n) \cap A$  are the so-called *coefficients of the semi-spray*.

**Definition 2.9.** We say that a semi-spray  $S$  is a *spray* if, in addition,  $A$  is conic, namely, if  $v \in A$ , then  $\lambda v \in A$  for every  $\lambda > 0$  and  $S$  has the following property:

If  $\beta = \dot{\alpha}$  is an integral curve of  $S$ , then  $\tilde{\beta}(t) = \lambda \dot{\alpha}(\lambda t)$   
is an integral curve of  $S$  for every  $\lambda > 0$ . (8)

Condition (8) is equivalent to the positive homogeneity of degree two of the functions  $G^i$  (recall (7)) on  $T\Omega \cap A$  (see [9, Chapter 4]). It is also equivalent to the equation  $[S, C] = C$ , where  $C$  is the canonical vertical vector field. We will say that the *geodesics of the spray* are the projections in  $M$  of the integral curves of  $S$

in  $A$ . In a system of coordinates  $(\Omega, \varphi)$ , a curve  $x : [a, b] \rightarrow \Omega \subset M$  is a geodesic of the spray  $S$  if and only if

$$\ddot{x}^i = -G^i \circ \tilde{\varphi}^{-1}(x, \dot{x}). \quad (9)$$

Moreover, a spray determines naturally an Ehresmann connection which is given by  $[S, \mathcal{V}TM]$ . A basis of the horizontal space is

$$-[S, \frac{\partial}{\partial y^i}] = \frac{\partial}{\partial x^i} - N_j^i(v) \frac{\partial}{\partial y^j},$$

where  $N_j^i(v) = \frac{1}{2} \frac{\partial G^i}{\partial y^j}(v)$  are called the non-linear coefficients. Having a horizontal subspace, we have a splitting of  $TTM$

$$TTM = \mathcal{V}TM + \mathcal{H}TM$$

and the vertical and horizontal projections. We can also define the non-linear covariant derivative of any vector  $X$  along a curve  $\alpha : [a, b] \rightarrow M$  as  $D_\alpha X = i_v^{-1}(\mathcal{V}\dot{X})$ . In a system of coordinates

$$D_\alpha X = \dot{X}^i \partial_i + \dot{\alpha}^i N_j^i(X) \partial_j,$$

where  $X^i$  are the coordinates of  $X$  in  $(\Omega, \varphi)$ . Then it follows straightforwardly that  $\gamma : [a, b] \rightarrow M$  is a geodesic of the spray if and only if  $D_\gamma \dot{\gamma} = 0$ , namely, the nonlinear covariant derivative of the velocity is zero.

#### 2.4. Anisotropic linear connections.

**Definition 2.10.** An *anisotropic linear connection* is a map

$$\nabla : A \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow T\mathcal{H}(M), \quad (v, X, Y) \mapsto \nabla_X^v Y := \nabla(v, X, Y) \in T_{\pi(v)}M$$

such that  $\nabla(v, X, \cdot)$  is an anisotropic derivation for every  $X \in \mathfrak{X}(M)$ , and this map is  $\mathcal{F}(M)$ -linear in  $X$ . Moreover, we will say that the anisotropic linear connection is *torsion-free* if

$$\nabla_X^v Y - \nabla_Y^v X = [X, Y]$$

for every  $X, Y \in \mathfrak{X}(M)$  and  $v \in A$ .

Given a system of coordinates  $(\Omega, \varphi)$ , we define the Chrystoffel symbols of  $\nabla$  as the functions  $\Gamma_{jk}^i : A \cap T\Omega \rightarrow \mathbb{R}$  satisfying that  $\nabla_{\partial_j}^v \partial_k = \Gamma_{jk}^i(v) \partial_i$ . Observe that a connection is free-torsion if and only if its Chrystoffel symbols are symmetric in  $j$  and  $k$ .

Given an anisotropic linear connection  $\nabla$ , let us defined for every  $A$ -admissible vector field  $V$  in  $\Omega \subset M$ , the tensors

$$P_V(X, Y, Z) = \frac{\partial}{\partial t} (\nabla_X^{V+tZ} Y) |_{t=0},$$

$$R^V(X, Y)Z = \nabla_X^V \nabla_Y^V Z - \nabla_Y^V \nabla_X^V Z - \nabla_{[X, Y]}^V Z$$

where  $X, Y, Z$  are arbitrary smooth vector fields on  $\Omega$ . Observe that  $P$  is an  $A$ -tensor, but  $R^V$  is not. The anisotropic tensor  $P$  will be called the *vertical derivative* of  $\nabla$  and the connection  $\nabla$  is said *Berwald* if and only if  $P = 0$ . Moreover, in a natural system of coordinates  $(T\Omega \subset A, \tilde{\varphi})$ ,

$$P_v(u, w, z) = u^i w^j z^k \frac{\partial \Gamma_{ij}^l}{\partial y^k}(v) \partial_l$$

for every  $u, w, z \in T_{\pi(v)}M$  and being  $u^i, w^i$  and  $z^i$  the coordiantes of  $u, w, z$ . Thus  $P$  is symmetric in the first two arguments if  $\nabla$  is torsion-free.

**Lemma 2.11.** *Let  $\nabla$  be an anisotropic linear connection. Then for any  $v \in A$  and  $X, Y, Z \in \mathfrak{X}(M)$ ,*

$$R_v(X, Y)Z := (R^V(X, Y)Z - P_V(Y, Z, \nabla_X^V V) + P_V(X, Z, \nabla_Y^V V))(\pi(v)), \quad (10)$$

where  $V$  is a vector field with  $V(\pi(v)) = v$ , is well-defined and  $R$  is an anisotropic tensor.

*Proof.* The only thing we have to prove is that the definition does not depend on the extension  $V$ . Let us fix an  $A$ -admissible vector field  $V$ , a system of coordinates in  $\Omega$  and denote the Chrystoffel symbols of  $\nabla^V$  as  $\tilde{\Gamma}_{ij}^k(p) = \Gamma_{ij}^k(V(p))$ . Then

$$R^V(X, Y)Z = \left[ Z^i Y^j X^p \frac{\partial \tilde{\Gamma}_{ij}^k}{\partial x^p}(q) - Z^i X^j Y^p \frac{\partial \tilde{\Gamma}_{ij}^k}{\partial x^p}(q) + Z^i Y^j X^m \left( \tilde{\Gamma}_{ij}^l(q) \tilde{\Gamma}_{lm}^k(q) - \tilde{\Gamma}_{im}^l(q) \tilde{\Gamma}_{lj}^k(q) \right) \right] \frac{\partial}{\partial x^k}. \quad (11)$$

Moreover,  $\frac{\partial \tilde{\Gamma}_{ij}^k}{\partial x^p}(q) = \frac{\partial \Gamma_{ij}^k}{\partial x^p}(V) + \frac{\partial V^l}{\partial x^p} \frac{\partial \Gamma_{ij}^k}{\partial y^l}(V)$  and  $(\nabla_X^V V)^k = X^p \frac{\partial V^k}{\partial x^p} + X^p V^l \Gamma_{pl}^k(V)$ . Then

$$R_V(X, Y)Z = \left[ Z^i Y^j X^p \left( \frac{\partial \Gamma_{ij}^k}{\partial x^p}(V) - V^l \Gamma_{pl}^h(V) \frac{\partial \Gamma_{ij}^k}{\partial y^h}(V) \right) + Z^i X^j Y^p \left( \frac{\partial \Gamma_{ij}^k}{\partial x^p}(V) - V^l \Gamma_{pl}^h(V) \frac{\partial \Gamma_{ij}^k}{\partial y^h}(V) \right) + Z^i Y^j X^m \left( \Gamma_{ij}^l(V) \Gamma_{lm}^k(V) - \Gamma_{im}^l(V) \Gamma_{lj}^k(V) \right) \right] \frac{\partial}{\partial x^k}, \quad (12)$$

from which follows the independence of  $R$  from the chosen extension  $V$ .  $\square$

**Definition 2.12.** Given an anisotropic linear connection  $\nabla$ , we say that the anisotropic tensor  $R$  defined in Lemma 2.11 is the curvature tensor of  $\nabla$ .

**Proposition 2.13.** *Let  $v \in A$  and  $u, w, z \in T_{\pi(v)}M$ . We have that  $R_v(u, w) = -R_v(w, u)$ . Moreover, if  $\nabla$  is torsion-free, then  $R$  satisfies the first Bianchi identity:*

$$R_v(u, w)z + R_v(w, u)z + R_v(z, u)w = 0.$$

*Proof.* It is a straightforward computation. First recall that  $R^V$  satisfies the first Bianchi Identity and then take into account that  $P$  is symmetric in the first two arguments as  $\nabla$  is torsion-free.  $\square$

**2.5. Covariant derivatives along curves.** In the following, given a smooth curve  $\gamma : [a, b] \rightarrow M$ ,  $\mathfrak{X}(\gamma)$  will denote the space of smooth vector fields along  $\gamma$  and  $\mathcal{F}(\gamma)$  the smooth real functions defined on  $[a, b]$ .

**Definition 2.14.** An anisotropic covariant derivation  $D_\gamma^v$  in  $A$  along a curve  $\gamma : [a, b] \rightarrow M$  is a map

$$D_\gamma^v : \mathfrak{X}(\gamma) \rightarrow T_{\pi(v)}M, \quad X \mapsto D_\gamma^v X$$

for every  $v \in A$  with  $\pi(v) = \gamma(t_0)$  such that

- (i)  $D_\gamma^v(X + Y) = D_\gamma^v X + D_\gamma^v Y$ ,  $X, Y \in \mathfrak{X}(\gamma)$ ,
- (ii)  $D_\gamma^v(fX) = \frac{df}{dt}(t_0)X(t_0) + f(t_0)D_\gamma^v X$  for any  $f \in \mathcal{F}(\gamma)$ ,  $X \in \mathfrak{X}(\gamma)$



- (iii)  $D_\gamma^V X(t) := D_\gamma^{V(t)} X$  is smooth for any  $V \in \mathfrak{X}(\gamma)$ ,  $X \in \mathfrak{X}(\gamma)$  and  $V$ ,  $A$ -admissible, that is,  $V(t) \in A$  for every  $t \in [a, b]$ .

**Proposition 2.15.** *Every anisotropic linear connection  $\nabla$  determines an induced anisotropic covariant derivative with the following property: if  $X \in \mathfrak{X}(M)$ , then  $D_\gamma^v X_\alpha = \nabla_\alpha^v X$ .*

*Proof.* It follows as in [8, Proposition 3.18]. Observe that in coordinates, the induced covariant connection is given by

$$D_\gamma^v X = \dot{X}^i(t_0) \frac{\partial}{\partial x^i} + \dot{X}^i(t_0) \dot{\gamma}^j(t_0) \Gamma_{ij}^k(v) \frac{\partial}{\partial x^k}. \quad (13)$$

□

**Definition 2.16.** We say that a curve  $\gamma : [a, b] \rightarrow M$  is a *geodesic* of the anisotropic linear connection  $\nabla$  if  $D_\gamma \dot{\gamma} = 0$ , where  $D_\gamma$  is the anisotropic covariant derivative associated with  $\nabla$ .

In coordinates, geodesics are given by the equation

$$\ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \Gamma_{ij}^k(\dot{\gamma}) = 0. \quad (14)$$

**Remark 2.17.** Every anisotropic linear connection  $\nabla$  determines a spray, which is given by the tangent vector to the curves  $\dot{\gamma} : [a, b] \rightarrow A$ , where  $\gamma$  is a geodesic of  $\nabla$ . The coefficients of the spray in a natural coordinate system  $(T\Omega, \tilde{\varphi})$  are given by  $G^i(v) = v^j v^k \Gamma_{jk}^i(v)$  for any  $v \in T\Omega \cap A$ . Observe that there can be several anisotropic connections that determine the same spray. In particular,  $\nabla$  and  $\tilde{\nabla}$  determines the same spray when the difference tensor, defined as

$$\mathfrak{L}_v(X, Y) = \nabla_X^v Y - \tilde{\nabla}_X^v Y, \quad (15)$$

for any  $X, Y \in \mathfrak{X}(M)$ , satisfies that  $\mathfrak{L}_v(v, v) = 0$  for every  $v \in A$ .

We say that a *two-parameter map* is a smooth map  $\Lambda : \mathcal{O} \rightarrow M$  such that  $\mathcal{O}$  is an open subset of  $\mathbb{R}^2$  satisfying the interval condition, namely, horizontal and vertical lines of  $\mathbb{R}^2$  intersect  $\mathcal{O}$  in intervals. We will use the following notation:

- (1) the  $t$ -parameter curve of  $\Lambda$  in  $s_0$  is the curve  $\gamma_{s_0}$  defined as  $t \rightarrow \gamma_{s_0}(t) = \Lambda(t, s_0)$
- (2) the  $s$  parameter curve of  $\Lambda$  in  $t_0$  is the curve  $\beta_{t_0}$  defined as  $s \rightarrow \beta_{t_0}(s) = \Lambda(t_0, s)$ .

Let us define  $\Lambda^*(TM)$  as the vector bundle over  $\mathcal{O}$  induced by  $\pi : TM \rightarrow M$  and  $\Lambda$ . Then we denote the subset of smooth sections of  $\Lambda^*(TM)$  as  $\mathfrak{X}(\Lambda)$ . Observe that a vector field  $V \in \mathfrak{X}(\Lambda)$  induces vector fields in  $\mathfrak{X}(\gamma_{s_0})$  and  $\mathfrak{X}(\beta_{t_0})$ . We can also define the *curvature operator* associated with an  $A$ -admissible two-parameter map  $\Lambda : [a, b] \times (-\varepsilon, \varepsilon) \rightarrow M$ ,  $(t, s) \rightarrow \Lambda(t, s)$ . Here  $A$ -admissible means that  $\dot{\gamma}_s(t) \in A$  for every  $(t, s) \in [a, b] \times (-\varepsilon, \varepsilon)$ . The curvature operator of  $\Lambda$  is a map  $R_\Lambda : \mathfrak{X}(\Lambda) \rightarrow \mathfrak{X}(\Lambda)$  defined, for any vector field  $W \in \mathfrak{X}(\Lambda)$ , as

$$R_\Lambda(W) := D_{\dot{\gamma}_s}^{\dot{\gamma}_s} D_{\dot{\beta}_t}^{\dot{\gamma}_s} W - D_{\dot{\beta}_t}^{\dot{\gamma}_s} D_{\dot{\gamma}_s}^{\dot{\gamma}_s} W - P(\dot{\beta}_t, W, D_{\dot{\gamma}_s}^{\dot{\gamma}_s} \dot{\gamma}_s) + P(\dot{\gamma}_s, W, D_{\dot{\gamma}_s}^{\dot{\gamma}_s} \dot{\beta}_t),$$

As in [4, Theorem 1.1], when the covariant derivative is induced by an anisotropic linear connection, it is possible to prove that

$$R_\Lambda(W) = R_{\dot{\gamma}_s}(\dot{\gamma}_s, \dot{\beta}_t)W,$$

where  $R_{\dot{\gamma}_s}$  is the curvature tensor associated with the anisotropic connection. Let us characterize geodesic variations with the curvature operator.

**Definition 2.18.** Given a geodesic  $\gamma$  of a spray  $S$ , we say that a vector field  $J$  along  $\gamma$  is a *Jacobi field* if it is the variational vector field of a variation of  $\gamma$  such that the longitudinal curves are geodesics.

**Proposition 2.19.** *Given a torsion-free anisotropic linear connection  $\nabla$  and a geodesic  $\gamma : [a, b] \rightarrow M$  of  $\nabla$ , if the vertical derivative of  $\nabla$  satisfies that*

$$P_v(v, v, u) = 0 \quad \forall v \in A \text{ and } u \in T_{\pi(v)}M, \quad (16)$$

*then Jacobi fields  $J$  along  $\gamma$  are determined by*

$$(D_{\dot{\gamma}})^2 J = R_{\dot{\gamma}}(\dot{\gamma}, J)\dot{\gamma}, \quad (17)$$

*where  $R$  is the curvature tensor of  $\nabla$ .*

*Proof.* It can be proved adapting the proof of [2, Prop. 8.1].  $\square$

**Definition 2.20.** Let  $\nabla$  be an anisotropic connection in  $A \subset M$  satisfying condition (16) and  $\gamma : [a, b] \rightarrow M$  a geodesic of  $\nabla$ . We say that the map

$$R_{\gamma} : \mathfrak{X}(\gamma) \rightarrow \mathfrak{X}(\gamma), \quad U \rightarrow R_{\gamma}(U) := R_{\dot{\gamma}}(\dot{\gamma}, U)\dot{\gamma} \quad (18)$$

*is the curvature operator of  $\gamma$ .*

**Proposition 2.21.** *Let  $\nabla$  be an anisotropic connection with vertical derivative satisfying (16). Then any other anisotropic connection  $\tilde{\nabla}$  with difference tensor (15) with respect to  $\nabla$  satisfying*

$$\mathfrak{L}_v(v, u) = 0, \quad \forall v \in A \text{ and } u \in T_{\pi(v)}M, \quad (19)$$

*has the same curvature operator as  $\nabla$ .*

*Proof.* Observe that for any vector field  $J$  along  $\gamma$ ,  $(\tilde{\nabla}_{\dot{\gamma}})^2 J = (\nabla_{\dot{\gamma}})^2 J$  because of condition (19). Moreover, a straightforward computation shows that the vertical derivative of the difference tensor satisfies that

$$\partial^\nu \mathfrak{L}_v(v, u, w) = -\mathfrak{L}_v(w, u) \quad (20)$$

(use that  $\mathfrak{L}_{v+tw}(v+tw, u) = 0$  for every  $t \in \mathbb{R}$  and apply derivation with respect to  $t$ ). If  $\tilde{P}$  is the vertical derivative of  $\tilde{\nabla}$ , then using (20), we get

$$\tilde{P}_v(v, v, u) = P_v(v, v, u) + \partial^\nu \mathfrak{L}_v(v, v, u) = -\mathfrak{L}_v(u, v) = 0.$$

This implies that (17) holds also for  $\tilde{\nabla}$ , then as the Jacobi fields are the same for both connections and we have already observed that  $(\tilde{\nabla}_{\dot{\gamma}})^2 J = (\nabla_{\dot{\gamma}})^2 J$ , we conclude that both connections have the same curvature operator.  $\square$

**2.6. Chern connection as Levi-Civita connection.** We say that a smooth, positive two-homogeneous function  $L : A \subset TM \setminus 0 \rightarrow \mathbb{R}$  is a pseudo-Finsler metric if  $A$  is a conic subset, namely, for every  $v \in A$  and  $\lambda > 0$  we have that  $\lambda v \in A$  and in addition the fundamental tensor defined as

$$g_v(u, w) := \frac{1}{2} \frac{\partial^2}{\partial t \partial s} L(v + tu + sw)|_{t=s=0} \quad (21)$$

for every  $v \in A$  and  $u, w \in T_{\pi(v)}M$ , is non-degenerate. Moreover, we define the Cartan tensor associated with  $L$  as

$$C_v(w_1, w_2, w_3) := \frac{1}{4} \frac{\partial^3}{\partial s_3 \partial s_2 \partial s_1} L \left( v + \sum_{i=1}^3 s_i w_i \right) \Big|_{s_1=s_2=s_3=0} \quad (22)$$

Observe that given any anisotropic linear connection we can compute the covariant derivative of a pseudo-Finsler metric as

$$\nabla_X^v(L) = X(L(V)) - \partial^\nu L(\nabla_X^V V)$$

taking into account that the vertical derivation of  $L$  satisfies  $\partial^\nu(L)(w) = 2g_v(v, w)$  (see for example [5, Proposition 2.3]), we conclude that

$$\nabla_X^v(L) = X(L(V)) - 2g_V(V, \nabla_X^V V).$$

The covariant derivative of its fundamental tensor can be computed as

$$\nabla_X^v(g)(Y, Z) = X(g_V(Y, Z)) - g_V(\nabla_X^V Y, Z) - g_V(Y, \nabla_X^V Z) - \partial^\nu g(\nabla_X^V V, Y, Z)$$

as the vertical derivation of  $g$  satisfies  $\partial^\nu g(X, Y, Z) = 2C_V(X, Y, Z)$ , then

$$\nabla_X^v(g)(Y, Z) = X(g_V(Y, Z)) - g_V(\nabla_X^V Y, Z) - g_V(Y, \nabla_X^V Z) - 2C_V(\nabla_X^V V, Y, Z).$$

This implies straightforward that the Chern connection (when interpreted as a family of affine connections) is the only anisotropic linear connection which is torsion-free and compatible with the metric in the sense that  $\nabla g = 0$ , namely, Chern connection is the Levi-Civita connection of the pseudo-Finsler metric (see [7], [9, Eqs. (7.20) and (7.21)] or [3, Proposition 2.3]).

**2.7. Berwald connection and the classical tensors.** We have seen in subsection 2.3 how to associate a non-linear connection with a spray. Moreover, geodesics can be recovered as the curves having parallel velocity with the non-linear connection, namely, in a natural coordinante system  $(\tilde{\varphi}, \Omega \times \mathbb{R}^n)$ , they satisfy

$$\ddot{x}^i = -\dot{x}^k N_k^i(\tilde{\varphi}^{-1}(x, \dot{x})),$$

since, by homogeneity, we can apply Euler's theorem to obtain that

$$y^k N_k^i(\tilde{\varphi}^{-1}(x, y)) = G^i(\tilde{\varphi}^{-1}(x, y)).$$

If we want to find an anisotropic connection with the same geodesics as the spray, then we need that

$$\dot{x}^i N_i^k(\tilde{\varphi}^{-1}(x, \dot{x})) = \dot{x}^i \dot{x}^j \Gamma_{ij}^k(\tilde{\varphi}^{-1}(x, \dot{x})).$$

Using again Euler's theorem, we obtain that

$$\Gamma_{ij}^k(\tilde{\varphi}^{-1}(x, y)) = \frac{\partial N_{ij}^k}{\partial y^j}(\tilde{\varphi}^{-1}(x, y)) = \frac{\partial^2 G^k}{\partial y^i \partial y^j}(\tilde{\varphi}^{-1}(x, y))$$

is a possible choice. These are the coefficients of the Berwald connection (see [9] and [2] for a treatment of the Berwald connection as an anisotropic linear connection). In the following, we will denote the Chern and Berwald connections, respectively, as  $\nabla$  and  $\tilde{\nabla}$ .

Let us now introduce some classical tensors. Recall that a pseudo-Finsler metric  $L : A \subset TM \setminus 0 \rightarrow \mathbb{R}$  determines a spray with

$$G^i = y^j y^k g^{is} \frac{1}{2} \left( \frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right), \quad (23)$$

for a system of natural coordinates  $(\tilde{\varphi}, \Omega \times \mathbb{R}^n)$ .

**Definition 2.22.** Let  $S$  be a spray in  $A \subset TM \setminus 0$ ,  $\tilde{\nabla}$  its Berwald connection,  $v \in A$ ,  $u, w, z \in T_{\pi(v)}M$  and  $V, X, Y$  and  $Z$  are arbitrary extensions of  $v, u, w$  and  $z$ . The *Berwald tensor*  $B$  is defined as the vertical derivative of the Berwald connection, namely,

$$B_v(u, w, z) = \frac{\partial}{\partial t} \left( \tilde{\nabla}_X^{V+tZ} Y \right) |_{t=0} \quad (24)$$

(see [9, Definition 6.1.1]). Assume now that  $(M, L)$  is a pseudo-Finsler manifold,  $S$  is the spray associated with it (see (23)) and  $\nabla$  the Chern connection associated with  $L$  (see §2.6). We define the *Chern tensor* of  $L$  as the vertical derivative of the Chern connection

$$P_v(u, w, z) = \frac{\partial}{\partial t} \left( \nabla_X^{V+tZ} Y \right) |_{t=0} \quad (25)$$

(see (7.23) in [9]). As  $\tilde{\nabla}$  and  $\nabla$  are torsion-free,  $B$  and  $P$  are symmetric in the first two arguments, and by homogeneity, it follows that  $B_v(v, u, w) = P_v(v, u, w) = 0$ .

Finally, we define the *Landsberg curvature* of a pseudo-Finsler metric  $L$  as

$$\mathfrak{L}_v(u, w, z) = g_v(B_v(u, w, z), v) \quad (26)$$

(see [9, Definition 6.2.1]).

Observe that if  $\nabla$  and  $\tilde{\nabla}$  are, respectively, the Chern and the Berwald connections, then

$$\mathfrak{L}_v(X, Y) = \nabla_X^v Y - \tilde{\nabla}_X^v Y. \quad (27)$$

Here  $\mathfrak{L}$  is determined by  $g_v(\mathfrak{L}_v(u, w), z) = \mathfrak{L}_v(u, w, z)$  (see (7.17) in [9]).

**Proposition 2.23.** *Given a pseudo-Finsler metric  $L$ , its Chern and Berwald connections determined the same Jacobi operator (17).*

*Proof.* Observe that the Berwald connection satisfies the hypotheses of Proposition 2.19. Moreover, the difference tensor between the Chern and Berwald connections is  $\mathfrak{L}$  (see (27)) and  $\mathfrak{L}_v(v, u) = 0$  for every  $v \in A$  and  $u \in T_{\pi(v)}M$ . Then we can apply Proposition 2.21.  $\square$

**2.8. Relation with linear connections in fiber bundles.** Given a linear connection  $\nabla$  in the fiber bundle  $\pi : \mathcal{V}TM \rightarrow A \subset TM$ , we can define an associated anisotropic connection as follows. Recall the map  $i_v$  in (5). We can define the vertical lift in  $v \in A$  of a vector  $u \in A$  as  $u^\mathcal{V} = i_v(u)$ . Moreover, we can also define the horizontal lift  $u^\mathcal{H}$  of a vector  $u \in TM$  in  $v \in A$  as the unique horizontal vector in  $T_v TM$  such that  $\pi_{TM}(u^\mathcal{H}) = u$ . In coordinates, if  $u = a^i \frac{\partial}{\partial x^i}$ , then

$$u^\mathcal{H} = a^i \left( \frac{\partial}{\partial x^i} - N_i^j(v) \frac{\partial}{\partial y^j} \right).$$

The anisotropic linear connection is defined as

$$\nabla_X^v Y = (\nabla_{X^\mathcal{H}} Y^\mathcal{V})(v).$$

It follows straightforwardly using the Christoffel symbols of the classical connections (see for example page 39 in [1]) that the Chern and the Cartan linear connections determine the same anisotropic linear connection as do Berwald and Hashiguchi ones. For an elegant proof of this fact see [10].

**2.9. Lie derivatives.** Given  $Z \in \mathfrak{X}(M)$ , the anisotropic Lie derivative  $\mathcal{L}_Z$  is the tensor derivation obtained in Theorem 2.7 when you consider the anisotropic derivation  $\delta^v(X) = [Z, X]$ , for every  $v \in TM \setminus 0$  and  $X \in \mathfrak{X}(M)$ . Therefore, we get the anisotropic Lie derivative tensor  $\mathcal{L}_Z T$  for every anisotropic tensor  $T$ . Let us see some particular cases. Let  $L : A \rightarrow \mathbb{R}$  be a pseudo-Finsler metric. Then for every  $X \in \mathfrak{X}(M)$ , we have

$$\mathcal{L}_X(L) = X(L(V)) - \partial^\nu(L)([X, V]),$$

where  $V$  is any vector field that extends  $v \in A$ . Now observe that  $\partial^\nu(L)(w) = 2g_v(v, w)$  (see for example [5, Proposition 2.3]). Then using Chern connection and that  $F(v)^2 = g_v(v, v)$  we get

$$\mathcal{L}_X(L) = 2g_V(\nabla_X^V V, V) - 2g_V(V, [X, V]) = 2g_v(\nabla_v^v X, v).$$

It follows that  $X$  is a Killing field of  $L$  if and only if  $\mathcal{L}_X L = 0$  and conformal if and only if  $\mathcal{L}_X L = fL$  for some function  $f : M \rightarrow \mathbb{R}$  (see for example [2, Proposition 6.1]). Moreover,

$$\mathcal{L}_X(g)_v(Y, Z) = X(g_V(Y, Z)) - g_V([X, Y], Z) - g_V(Y, [X, Z]) - 2C_V([X, V], Y, Z)$$

since  $\partial^\nu g = 2C$ . Using the Chern connection we get the tensorial expression

$$\mathcal{L}_X(g)_v(u, w) = g_v(\nabla_u^v X, w) + g_v(u, \nabla_w^v X) - 2C_v(\nabla_v^v X, u, w),$$

which gives another characterization of Killing fields:

$$g_v(\nabla_u^v X, w) + g_v(u, \nabla_w^v X) - 2C_v(\nabla_v^v X, u, w) = 0,$$

for every  $v \in A$  and  $u, w \in T_{\pi(v)}M$ . Let us observe that given a diffeomorphism  $\psi : M \rightarrow M$ , we can define the pullback  $\psi^*(T)$  of an anisotropic tensor  $T$  as the anisotropic tensor given by  $\psi^*(T)_v = T_{\psi^*(v)}$ , where  $\psi^*$  is the differential of  $\psi$ .

**Proposition 2.24.** *If  $X \in \mathfrak{X}(M)$  and  $T \in \mathfrak{T}_s^0(M, A)$ , then*

$$\mathcal{L}_X T = \lim_{t \rightarrow 0} \frac{1}{t} (\psi_t^*(T) - T),$$

where  $\psi_t$  is the (possibly local) flow of  $X$ .

*Proof.* It follows the same steps as in the classical isotropic case (see for example [8, Proposition 9.21] excepting for the following:

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{1}{t} (T_{\psi_t^*(Y_p)}(V_{\psi_t(p)}, W_{\psi_t(p)}) - T_{Y_p}(V_p, W_p)) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (T_{\psi_t^*(Y_p)}(V_{\psi_t(p)}, W_{\psi_t(p)}) - T_{Y(\psi_t(p))}(V_{\psi_t(p)}, W_{\psi_t(p)})) \\ & \quad + \lim_{t \rightarrow 0} \frac{1}{t} (T_{Y(\psi_t(p))}(V_{\psi_t(p)}, W_{\psi_t(p)}) - T_Y(V_p, W_p)) \\ &= \partial^\nu T_{Y_p}(\lim_{t \rightarrow 0} \frac{1}{t} (\psi_t^*(Y_p) - Y(\psi_t(p))), V_p, W_p) + X(T_Y(V, W))(p) \end{aligned}$$

In the last equality, we have used that if a function  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is  $C^1$  and  $v, w : (-\epsilon, \epsilon) \rightarrow \Omega$  are also  $C^1$  with  $\lim_{t \rightarrow 0} v(t) = \lim_{t \rightarrow 0} w(t) = p$ , then  $\lim_{t \rightarrow 0} \frac{1}{t} (f(v(t)) - f(w(t))) = df_p(\lim_{t \rightarrow 0} \frac{1}{t} (v(t) - w(t)))$  (apply, for example, the mean value theorem). Finally observe that

$$\lim_{t \rightarrow 0} \frac{1}{t} (\psi_t^*(Y_p) - Y(\psi_t(p))) = \lim_{t \rightarrow 0} \psi_t^*(Y_p - \psi_{-t}^*(Y(\psi_t(p)))) = -[X, Y]_p,$$

where we have used [8, Proposition 1.58]. □

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